Generate your own logarithm tables

To really understand logarithms, one should generate their own tables so that they can understand round off errors and techniques to improve the speed and accuracy of calculations. This also gives us an opportunity to understand the theory by applying it. The rules for exponents are as follows:

$A^m x A^n = A^{m+n}$ and $(A^m)^n = A^{mxn}$

We can prove them as follows:

$A^3 = AxAxA$ $A^2 = AxA$ then $A^3A^2 = AxAxA\ x\ AxA = A^{3+2}$

$(A^3)^2 = A^3A^3 = A3+3 = A2x3$ We used the above proof to prove this one.

We would like to find to what power we must raise 10 to get the number 2. We begin by constructing a table of successive square roots of 10:

Dec. Value of power	**power**	**Value of 10 raised to that power**
1	**1**	**10**
.5	1/2	3.1622776
.25	1/4	1.7782794*
.125	1/8	1.3335214
.0625	1/16	1.1547819
.03125	1/32	1.0746078*
.015625	1/64	1.0366329*
.0078125	1/128	1.0181517
.00390625	1/256	1.0090350*
.001953125	1/512	1.0045073
.0009765625	1/1024	1.0022511
.00048828125	1/2048	1.0011249
.000244140625	1/4096	1.0005623*

We now start to divide by the largest value in the last column that does not exceed 2. We then divide that quotient by the largest value in the last column that does not exceed that quotient. We continue to repeat the process until the quotient is very close to 1:

2/1.7782794=1.1246826	.25
1.1246826/1.0746078=1.0465982	.03125
1.0465982/1.0366329=1.0096131	.015625
1.0096131/1.0090350=1.0005729	.00390625
1.0005729/1.0005623=1.0000105	<u>.000244140625</u>
$\log_{10}(2)$=	.301025390625

The $\log_{10}(2)$ is the sum of the powers of 1/2 corresponding to the numbers in the third column used as divisors. We can make the results more accurate by doing more divisions to find more numbers to sum. However, we would have to use numbers of greater accuracy of our powers of ten. If we look at the answer as follows:

$$\log^{10}(2)=10^{1/4+1/32+1/64+1/256+1/4098}=10^{1/4}10^{1/32}10^{1/64}10^{1/256}10^{1/4098}\ldots$$

We note that we have applied the rule for multiplying numbers raised to a power. In taking the roots of powers of 2, we were applying the rule for raising a number to a power: 1/4=1/2 x 1/2.

To find the $\log_{10}(3)=.47712$, we would have to take three more square roots and used more digits in our powers of ten. The answer is $10^{1/4+1/8+1/16+1/32+1/128+1/2048+1/16384}$. This becomes to difficult a task for one person and would require lots of checking to make sure we had the correct answer.

But let us say that we use log10(2)=.3010 and log10(3)=.4771, we could calculate the other logarithms using the rules for logarithms and some approximations. Let us prove the rules for logarithms using the definition of a logarithm as $A \wedge \log_A(b)=B$.

$$\begin{aligned} AB&=10 \wedge \log_{10}(AB) \\ &=10 \wedge \log_{10}(A) \times 10 \wedge \log_{10}(B) \\ &=10 \wedge (\log_{10}(A) + \log_{10}(B)) \quad \text{using the rule of exponents} \end{aligned}$$

Equating exponents: $\log_{10}(AB)=\log_{10}(A) + \log_{10}(B)$. Then we have:

$$\begin{aligned} A^n&=10 \wedge \log_{10}(A^n) \\ &=(10 \wedge \log_{10}(A))^n = 10 \wedge n \log_{10}(A) \end{aligned}$$

Equating exponents: $\log_{10}(A^n) = n \log_{10}(A)$

Some extras: $10 \wedge \log_{10}(10) =10^1$, $\log_{10}(10)=1$ and $10 \wedge \log_{10}(1)=1=10^0$ $\log_{10}(1)=0$

We can now use these rules:

number	factors	log	Ans
1			0
2			.3010
3			.4771
4	2^2	2log(2)	.6020
5	10/2	Log(10)-log(2)	.6990
6	2x3	Log(2)+log(3)	.7781
7			
8	2^3	3log(2)	.9030
9	3^2	2log(3)	.9542
10	2x5	1	1.0000
48	2^4x3	4log(2)+log(3)	1.6811
50	5x10	Log(5)+log(10)	1.6990

Log(49)=log(7^2)=2log(7) ≈ (log(48)+log(50))/2 =(1.6811+1.6990)/2=3.3801/2=1.6900
Log(7)=1.6900/2=.8450

To find the log(11), we use log(99)≈(log(100)+log(98))/2 and for log(13), log(65)=(log(66)+log(64))/2
After that, we can just take the average of the numbers on either side of the prime number.

We would like to get the numbers to two decimal places. We do this by going from 101 to 999 and subtracting 2 (dividing by 100).

number	factors	log	Ans		New range
101		(log(110)+log(108))/2	2.0043	.0043	**1.01**
102	2x51	Log(2)+log(51)	2.0086	.0086	**1.02**
103		(log(110)+log(108))/2	2.0128	.0128	**1.03**
104	2^3x13	3log(2)+log(13)	2.0170	.0170	**1.04**
105	3x5x7	Log(15)+log(7)	2.0211	.0211	**1.05**
106	2x53	Log(2)+log(53)	2.0253	.0253	**1.06**
107		(log(110)+log(108))/2	2.2093	.2093	**1.07**
108	3^3x4	3log(3)+2log(2)	2.0334	.0334	**1.08**
109		(log(110)+log(108))/2	2.0374	.0374	**1.09**
110	2x5x11	1	2.0413	.0413	**1.10**

To complete the process, we round the numbers up to the third decimal place. We will have a log table with nearly 900 numbers.

Log tables of five digits

This is a simple process that one can do in a few hours. However for more serious calculations, we need log tables that can give us five digit answers. To do this we will show the process.
We start with the following formula that can be derived using integral calculus:

$$A=\ln((1+y)/(1-y))=2y(1+y^2/3+y^4/5+y^6/7+y^8/9+y^{10}/11+y^{12}/13+. . .)$$

To make the infinite series converge rapidly, we need to use small values for y. We go back to our first table where we where listing the prime factors for each number. We are going to use simultaneous equations to help us find the logs of 2 and 3. Since 6=2x3 and 8=2^3 we use the reciprocal of the number between them y=1/7.
Then A=ln(8/6)=ln(4/3)=2ln(2)-ln(3)=2/7(1+1/147+1/12005+ 1/823543+ =.2876820668
We nee another equation. So we use y=1/17 since 16=2^4 and 8=2x3^2. This gives us
B=ln(18/16)=ln(9/8)=2ln(3)-3ln(2)=2/17(1+1/867+1/417605 =.1177830349 You will notice that with a smaller number (bigger reciprocal), the series converges faster.

$2\ln(2)-\ln(3)=A$
$-3\ln(2)+2\log(3)=B$
$\ln(2)=2A+B = .6931471$
$\ln(3)=3A+2B=1.0986122$

Now we need to find the ln(10).
$Y=19$
$C=\ln(20/18)=\ln(10/9)=\ln(10)-2\log(3)=2/19(1+1/1083+1/651605+ =.1053605153$
$Ln(10)=2\ln(3)+C=2.302585$

We need another formula.

$$A \wedge \log_A(B)= B$$
$$\log_A(B)\log_B(A)=\ln_B(B)=1 \text{ then } \log_A(B)= 1/\log^B(A)$$
$$A \wedge \log_A(C)=C$$
$$\log_A(C) \log_B(A)=\log_B(C)$$
$$\log_A(C)= (1/\log_B(A))\log_B(C) = \log_A(B) \log_B(C)$$

In our case A=e=2.718281.. And B=10
$Log_{10}(2)=\log_{10}(e)loge(2)= \ln(2)/\ln(10)=\ln(2)/2.302585=.3010299$
Similarly, $Log(3)=\ln(3)/2.302585=.4771212$

This was less work to find the logs of 2 and 3 than the previous approach, but we had to know how to solve simultaneous equations and use integral calculus. We can no longer use the approximation approach to find the logs of the prime numbers if we want to maintain accuracy. We can find ln(5) by the following: $\log(5)=\log(10/2)=\log(10)-\log(2)=1-\log(2)=.6989700$
For log(7) we do the following:
$Y=29$
$D=2/29(1+1/2523+1/3536405+)=.068992671$
$Ln(7)=\ln(5)+\log(3)-\ln(2)-D=1.945910$
$Log(7)=\ln(7)/2.302585-.8450980$

We do not want to continue going to our factor table, so we use the following formula from differential calculus:
$Log(x+x_0)=\log(x_0)+\ln(10)(x/x_0-(x/x_0)^2/2+(x/x_0)^3/3+. . .)$
Let $x=-1$ and $x_0=100$
$Log(99)=2-(1/100+1/20000+1/3000000+)/2.302585=1.9956351$
$Log(11)=\log(99)-2x\log(3)=1.041392$

For 13, we have $\log(7x13)=\log(1+90)=\log(90)+\ln(10)(1/90-(1/90)^2/2+(1/90)^3/3–(1/90)^4/4+...)$
$=\log(90)+ 0.011049836/2.302585=\log(90)+0.004798883$
$\log(13)=\log(10)+2\log(3)-\log(7)+0.004772473574=1.113943352$
If you choose, you can now generate your own 5 digit logarithm table.

Logarithms of more than five digit accuracy

If we would like to still get even more accurate values for our low prime numbers, we can use bigger reciprocals and use more simultaneous equations. If we use the reciprocals of 71, 97, 99, and 127 we get the following:

71	2/71(1+1/15123+1/127058405	=.0281 708	=A=.02817087696666490
97	2/97(1+1/28227+1/442646405	=.0206 1928	=B=.02061928720273214
99	2/99(1+1/29403+1/480298005	=.0202 0270	=C=.02020270731751638
127	2/127(1+1/48387+1/1300723205	=.0157 4835	=D=.015748356968138632

This gives us the following equations:

$2\ln(2)+2\ln(3)-\ln(5)-\ln(7)=A$
$4\ln(2)+\ln(3)\qquad +2\ln(7)=B$
$\ln(2)\qquad 2\ln(5)-2\ln(7)=C$
$6\ln(2)-2\ln(3)\qquad -\ln 7=D$

We will useinverted matrices to solve these equations:

$$\begin{pmatrix} 2 & 2 & -1 & -1 \\ -4 & -1 & 0 & 2 \\ 1 & 0 & 2 & -2 \\ 6 & -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} \ln(2) \\ \ln(3) \\ \ln(4) \\ \ln(5) \end{pmatrix} = \begin{pmatrix} (A) \\ (B) \\ (C) \\ (D) \end{pmatrix}$$

Determinant of the matrix is

$$\begin{vmatrix} 2 & 2 & -1 & -1 \\ -4 & -1 & 0 & 2 \\ 1 & 0 & 2 & -2 \\ 6 & -2 & 0 & -1 \end{vmatrix} = -2 \text{ x } \begin{vmatrix} -4 & 0 & 2 \\ 1 & 2 & -2 \\ 6 & 0 & -1 \end{vmatrix} - \begin{vmatrix} 2 & -1 & -1 \\ 1 & 2 & -2 \\ 6 & 0 & -1 \end{vmatrix} -2 \text{ x } \begin{vmatrix} 2 & -1 & -1 \\ -4 & 0 & 2 \\ 1 & 2 & -2 \end{vmatrix} = -2\text{x} -16 - 19 -2\text{x } 6 = 1$$

The inverse matrix is on the next page.

$$
\left(
\begin{array}{c}
\begin{vmatrix} -1 & 0 & 2 \\ 0 & 2 & -2 \\ -2 & 0 & -1 \end{vmatrix}
-
\begin{vmatrix} 2 & -1 & -1 \\ 0 & 2 & -2 \\ -2 & 0 & -1 \end{vmatrix}
\quad
\begin{vmatrix} 2 & -1 & -1 \\ -1 & 0 & 2 \\ -2 & 0 & -1 \end{vmatrix}
-
\begin{vmatrix} 2 & -1 & -1 \\ -1 & 0 & 2 \\ 0 & 2 & -2 \end{vmatrix}
\\[1em]
-
\begin{vmatrix} -4 & 0 & 2 \\ 1 & 2 & -2 \\ 6 & 0 & -1 \end{vmatrix}
\quad
\begin{vmatrix} 2 & -1 & -1 \\ 1 & 2 & -2 \\ 6 & 0 & -1 \end{vmatrix}
-
\begin{vmatrix} 2 & -1 & -1 \\ -1 & 0 & 2 \\ 6 & 0 & -1 \end{vmatrix}
\quad
\begin{vmatrix} 2 & -1 & -1 \\ 4 & 0 & 2 \\ 1 & 2 & -2 \end{vmatrix}
\\[1em]
\begin{vmatrix} -4 & -1 & 2 \\ 1 & 0 & -2 \\ 6 & -2 & -1 \end{vmatrix}
-
\begin{vmatrix} 2 & 2 & -1 \\ 1 & 0 & -2 \\ 6 & -2 & -1 \end{vmatrix}
\quad
\begin{vmatrix} 2 & 2 & -1 \\ -4 & -1 & 2 \\ 6 & -2 & -1 \end{vmatrix}
-
\begin{vmatrix} 2 & 2 & -1 \\ -4 & -1 & 2 \\ 1 & 0 & -2 \end{vmatrix}
\\[1em]
-
\begin{vmatrix} -1 & -1 & 0 \\ 1 & 0 & 2 \\ 6 & -2 & 0 \end{vmatrix}
\quad
\begin{vmatrix} 2 & 2 & -1 \\ 1 & 0 & 2 \\ 6 & -2 & 0 \end{vmatrix}
-
\begin{vmatrix} 2 & 2 & -1 \\ -4 & -1 & 0 \\ 6 & -2 & 0 \end{vmatrix}
\quad
\begin{vmatrix} 2 & 2 & -1 \\ -4 & -1 & 0 \\ -2 & 0 & -1 \end{vmatrix}
\end{array}
\right)
=
\begin{pmatrix} 10 & 12 & 5 & 4 \\ 16 & 19 & 8 & 6 \\ 23 & 28 & 12 & 9 \\ 28 & 34 & 14 & 11 \end{pmatrix}
$$

ln(2)=10A+12B+5C+4D=0.693147180559
ln(3)=16A+19B+8C+6D=1.09861228866
ln(5)=23A+28B+12C+9D=1.60943791243
ln(7)=28A+34B+14+11D=1.94591014905
ln(10)=ln(2)+ln(5) =2.3025850929

With all of this work, we would use a different approach if we were doing this calculations on a computer.