Chapter IV – Exponents and Logarithms

A. Introduction

Starting with addition and defining the notations for subtraction, multiplication and division, we discovered negative numbers and fractions. With defining new notations built upon older notations, we were able to develop a set of rules for performing the basic operations with these newly defined numbers. We developed an understanding and facility with these numbers and operations by practicing sample problems with them. In the traditional treatment of exponents and their inverses — logarithms, we do not practice the operations directly, but look results up in tables. This approach masks the full comprehension in a manner similar to a way the calculator does. Could you give an approximate answer to $2^{3.33}$? One of the problems you may observe is that up to exponents, all of our problems have involved integers or rational numbers.

B. Definition of an exponential

A number raised to a power is defined as the number of times you multiply that number times one. Thus $2^3 = 1 \times 2 \times 2 \times 2 = 8$ You will notice that we used a positional notation rather than introduce a new operator. In programming a double asterisk is used, and in some texts a caret is used. An example is given as follows: 2^3=2**3 = 2^3

Let us look at 2^3 and 3^2: $2^3 = 1 \times 2 \times 2 \times 2 = 8$ $3^2 = 1 \times 3 \times 3 = 9$. The exponential operation is not commutative as are addition and multiplication. The order of operands does make a difference in the answer.

What value of A will give the same value for the base? $2^A = 2 = 1 \times 2$ The value of A=1.

What value of B will give the same exponent? $B^2=2$ It would have to be a number ($2^{1/2}$) that when multiplied by itself would give us 2. If we used 3 as the exponent, we would get a different answer ($3^{1/3}$). Thus only with the exponent, do we have a unique identity element.

C. Properties of the exponent

There are several interesting operations to study with exponents.
$a^2 \times a^3 = a \times a \times a \;\; \times \;\; a \times a = a^{(3+2)}$
When we multiply like bases raised to a power, we add the exponents.

$a^5 / a^3 = \; a \times a \;\; \times \; a \times a \times a /(a \times a \times a) = a \times a = a^{(5-3)} = a^2$
When we divide numbers with like bases, we subtract the exponents.

$(a^2)^3 = a^2 \times a^2 \times a^2 = a^{3\times 2}$
When we raise a base that has already be raised to a power, we multiply the exponents.

Look at the following: $(a^2b^3)^5 = a^2b^3 \times a^2b^3 \times a^2b^3 \times a^2b^3 \times a^2b^3 \times a^2b^3 = a^2a^2a^2a^2a^2 \times b^2b^2b^2b^2b^2$
$= a^{2\times 5}b^{3\times 5}$

By going to the fundamental definition of exponents, we find that we can derive special relationships. Let us use our existing formula to develop new relationships.

$a^0 = a^{1-1} = a^1 / a^1 = a/a = 1$

$a^{-3} = a^{0-3} = a^0/a^3 = 1/a^3$

The secret to learning exponents is to practice deriving the formula with numbers and with algebraic symbols. Let us summarized what we have learned:

$a^3 = = 1 \times a \times a \times a = a \times a \times a$	Definition
$a^b a^c = a^{b+c}$	Add exponents when multiplying with like bases
$a^b / a^c = a^{b-c}$	Subtract exponents when dividing like bases
$a^0 = 1$	A number raise to 0 is 1
$a^1 = a$	One is the identity element
$a^{-b} = 1/a^b$	a negative exponent inverts
$(a^b)^c = a^{b \times c}$	Exponents of exponents are mult.
$(a^b c^d)^e = a^{be} c^{de}$	

D. Numerical examples

The best way to understand new concepts is to use numerical examples.
Let us evaluate $16^{2.75} = 16^{2\,3/4}$. We can write this as $16^{2+1/4+1/2}$ = using the expanded notational equivalent of a number. Using the multiplicative/summation rule for exponents, this gives us $16^2 \times 16^{1/2} \times 16^{1/4}$. Now using our knowledge of roots we get 256 x4 x 2 =1024.
For $4^{.75}$, we get $4^{1/2+1/4} = 4^{1/2} \times 4^{1/4} = (2^2)^{1/2} \times (2^2)^{1/4} = 2^{2 \times 1/2} \times 2^{2 \times 1/4} = 2^1 \times 2^{1/2} = 2 \times 1.414..$ $= 2.828...$

E. The inverses

Only the exponent had a unity operator. Thus, $a^1 = a$, so if we have a^3, then what do we have to do to the 3 to make it a 1? We multiply it by 1/3. $a^1 = a^{3 \times 1/3} = (a^3)^{1/3}$.

In an example, $2^3 = 8$ Then $8^{1/3} = (2^3)^{1/3} = 2^{3 \times 1/3} = 2^1 = 2$. This inverse has now allowed us to use a fraction as an exponent completing the use of previously defined elements (positive and negative integers and fractions which can also be positive and negative). Suppose that we wanted to find $2^{1/3}$. We would have to find a number such that when we multiplied it by itself three times would get the number 2. $2^{1/3} = (a \times a \times a)^{1/3} = a$. We will get the answer by a guessing procedure.

Since $1 \times 1 \times 1 = 1 < 2 < 2 \times 2 \times 2 = 8$, we guess that the number must lie between 1 and 2.

1. Finding cube root of 2

Since	1.5 x 1.5 x1.5 = 3.375	1	< a < 1.5
Since	1.2 x 1.2 x 1.2 = 1.728	1.2	< a < 1.5
Since	1.3 x 1.3 x 1.3 = 2.197	1.2	< a < 1.3
Since	1.25 x 1.25 x 1.25 =1.953..	1.25	< a <1.3
Since	1.27 x 1.27 x 1.27 =2.048..	1.25	< a < 1.27
Since	1.26 x1.26 x1.26=2.0003..	1.25	< a <1.26
Since	1.259x1.259x1.259=1.995..	1.259	< a <1.26

After seven guesses, we have found the answer to four significant digits. This is a very time consuming process and gives us an incentive to look for a better way. What is not obvious is that we would have to continue this process indefinitely because we could not find a repeating pattern that would indicate that the answer is the ratio (**ratio**nal number) of two integers. Thus we have discovered a new number — an ir**ratio**nal one, one that can not be represented by the ratio of two numbers. However we can approximate the answer by a ratio as we have done with the decimal 1.259 = 1+ 259/1000. If we had continued the guessing process, we would have found that the answer is greater than 1.259921..

2. Approximations and significant figures

Since irrational numbers are never ending we indicate the is as follows: $2^{1/2} = 1.414...$ Sometimes we leave off the ... , and approximate the number as 1.414   The number 1.14146... would be approximated as follows: 1.415 We will find that we can no longer write an exact answer and in some cases there may be so many digits we would be satisfied with just a few:

$$234.567 = 2.35\ 10^2$$

We have to experiment to see how many digits we must utilize to get the answer correct to 3, 4 , or perhaps 5 significant figures.

3. Finding $\log_2(10)$

Let us now rephrase our inverse question. To what power do we have to raise 2, to get 10 for the answer.

$$2^a = 10$$

Since 2^3=8 and 2^4=16, we know $3 < a < 4$.
If a =3.5, then $2^{3.5} = 2^{3+.5} = 2^3 \times 2^{.5} = 8\text{x } 2^{.5}$ where we have made use of the rules for exponents. We have made a bold assumption that non-integers obey the same rules as integers. With the guessing process, we find that $2^{.5}$=1.414, and the we calculate 8x1.414 =11.3. Thus $3 < a < 3.5$. Our next guess would be a = 3.2 which gives us 8 x $2^{.2}$=8x1.148..=9.18 For this we first find the tenth root of 2 and multiply the result twice. Since $2^{.1}$ =1.0717.., 1.0717x1.0717=1.148..
Calculating the hundredth root gets somewhat onerous.

Since taking the square root of a number is relatively easy, we could try to get the answer by using successive square roots of 2 as described in chapter II.

$2^{1/2} = 1.4142136$
$2^{1/4} = 1.1892071$
$2^{1/8} = 1.0905077$
$2^{1/16} = 1.0442738$

We start by dividing 10 by 2^3=8, giving us 1.25
We can divide the 1.25 by $2^{1/4}$ giving us 1.0511205
We can divide 1.0511205 by $2^{1/16}$ instead of $2^{1/8}$ for $2^{1/8}$ is greater than 1.0511205

Thus $10 = 2^3 \times 2^{1/4} \times 2^{1/16} = 2^{3.31..}$ (Multiplication is the inverse of division)
$= 8 \times 1.1892071 \times 1.0442738 = 9.93$ (not quite 10, but close enough)

With more calculations, we would have found that a =3.321928.. We used the rule for the product of numbers with the same base to add the exponents. We also used a binary number approach to find the logarithm for which we were searching.

4. The logarithm

We now invent a special function called the log function which strips off the exponent.

$\log_2(8) = \log_2(2^3) = 3$ $\log_{\mathbf{2}}(\mathbf{2}^3)$ (notice the same numbers (2))
Thus $\log_2(10) = \log_2(2^{3.321928}) = 3.321928$
Since we wrote $10=2^a$, $\log_2(10)= a$. Rewriting we have $10 = 2^a = 2^{\log 2(10)}$.
In another example: $16^{2\,3/4} = 2^{4\times(2\,3/4)} = 2^{11}$ Thus $\log_2(16^{2\,3/4}) = 11$.

F. Logarithms

We have defined a logarithm as a function which strips off the exponent.

$$a = b^{\log b(a)}$$

We can develop the following 7 rules for logarithms

1. $\log_b(a \times c) = \log_b(b^{\log b(a)} b^{\log b(c)}) = \log_b(b^{\log b(a) + \log b(c)}) = \log_b(a) + \log_b(c)$
2. $\log_b(a / c) = \log_b(b^{\log b(a)} / b^{\log b(c)}) = \log_b(b^{\log b(a) - \log b(c)}) = \log_b(a) - \log_b(c)$
3. $\log_b(a^c) = \log_b((b^{\log b(a)})^c) = \log_b((b^{c \log b(a)})) = c \log_b(a)$
4. $\log_b(1) = \log_b(b^0) = 0$
5. $\log_b(b) = \log_b(b^1) = 1$

Let $a = b1^{\log b1(a)}$

6. $\log_{b2}(a) = \log_{b2}(b1^{\log b1(a)})$
$= \log_{b1}(a) \log_{b2}(b1)$
$= \log_{b2}(b1) \times \log_{b1}(a)$

Now let $b2 = b1^{\log b1(b2)}$
$\log_{b2}(b2) = 1 = \log_{b2}(b1) \log_{b1}(b2)$

7. $\log_{b2}(b1) = 1/\log_{b1}(b2)$

From first principles, we are able to develop the seven rules for logarithms. In the previous chapter, we saw how to find the logarithm by a guessing procedure.

G. Calculations

To calculate the logarithm of a number is a three step process. The steps are as follows: calculate a powers of two table for the logarithmic base of interest, divide the number by successive powers of two, and then add these powers of two to calculate the value of the logarithm. A fourth step is to check the results. Let us start by finding the $\log_2(10)$

1. Step 1– powers of two table

Power of 2	**value**	**(Table for finding logs in base 2)**			
1/2	1.4142135623	1/32	1.0218971486	1/512	1.0013547198
1/4	1.1892071150	1/64	1.0108892860	1/1024	1.0006771306
1/8	1.0905077326	1/128	1.0054299011	1/2048	1.0003385080
1/16	1.0442737824	1/256	1.0027112750		

2. Step 2 – successive divides

$2^3 < 10 < 2^4$

		root	decimal value
$10 / 2^3 = 10/8$	=1.25	3	3
1.25 / 1.1892071150	=1.0511205190	1/4	.25
1.0511205190 / 1.0442737824	=1.0065564574	1/16	.0625
1.0065564574 / 1.0054299011	=1.0011204723	1/128	.0078125
1.0011204723 / 1.0006771306	=1.0004430417	1/1024	.0009765635
1.0004430417 / 1.0003385080	=1.0001044983	1/2048	.00048828175

3. Step 3 – add the powers of 2

$\log_2(10) = 3+1/4 +1/16 + 1/128 + 1/1024 +1/2048 + \ldots = 3\ 659/2048 +\ldots = 3.321\ldots$

4. Step 4 — check calculations

$$2^{3.217} = 2^{3+1/4+1/16+1/128+1/1024+1/2048}$$
$$= 2^3 \times 2^{1/4} \times 2^{1/16} \times 2^{1/128} \times 2^{1/1024} \times 2^{1/2048} \times \ldots$$
$$= 8 \times 1.189207 \times 1.044273 \times 1.005429 \times 1.000677 \times 1.000338 \times \ldots$$
$$= 10$$

Let us now try finding $\log_{10}(2)$.

1. Step 1– powers of two table

Power of 10	**Value**	**(Table for finding logs in base 10)**			
1/2	3.16227	1/32	1.07460	1/512	1.00450
1/4	1.77827	1/64	1.03663	1/1024	1.00225
1/8	1.33352	1/128	1.01815	1/2048	1.00112
1/16	1.15478	1/256	1.00903	1/4096	1.00056

2. Step 2 – successive divides
$10^0 < 2 < 10^1$

	root	decimal value
2 / 1.77827 = 1.12468	1/4	.25
1.12468 / 1.07460 =1.04661	1/32	.03125
1.04661 / 1.03663 =1.00962	1/64	.015625
1.00962 / 1.00903 =1.00059	1/256	.00390625
1.00059 / 1.00056 =1.00003	1/4096	.000244140

3. Step 3 – add the powers of 2
$\log_{10}(2) = 1/4 + 1/32 + 1/64 + 1/256 + 1/4096 = .30102$
4. Step 4 — check calculations
$10^{.30102...} = 10^{1/4 + 1/32 + 1/64 + 1/256 + 1/4096 + ...}$
$= 10^{1/4} \times 10^{1/32} \times 10^{1/64} \times 10^{1/256} \times 10^{1/4096} \times ...$
$= 1.77827 \times 1.07460 \times 1.03663 \times 1.00903 \times 1.00056$
$= 2$
If we had calculated powers of ten then the check calculation would be as follows:
$10^{.30102..} = 10^{.3} \times 10^{.001} \times 10^{.00002}$
$= 1.9952 \times 1.0023 \times 1.00004 = 1.99986..$
Note that $\log_2(10)\log_{10}(2) = 1 = 3.3219 \times .30102 = .999999...$ (almost 1)

Finding values of numbers raised to a power

Let us now find $324^{10.3}$.
$324 = 1.265625 \times 2^8$
$324^{.3} = 1.265625^{.3} \times 2^{2.4} = 1.265625^{.3} \times 2^{.4} \times 2^2$

Convert to binary
.3 = .01001100
=.25 + .03125 +.015625 =.296875

.4=.011001100
=.25+.125 +.015625 + .0078125=.3984375

1.265625

Power of 2	**value**
1/4	1.0606602
1/32	1.0073886
1/64	1.0036875

2.
(Table for finding logs in base 2)

1/4	1.1892071
1/8	1.0905077
1/64	1.0108893
1/128	1.0054299

$1.26525^{.296875} = 1.0606602 \times 1.0073886 \times 1.0036875 = 1.0724371$
$2^{..3984375} = 1.1892071 \times 1.0905077 \times 1.0108893 \times 1.0054299 = 1.3180796$

$1.26525^{.296875} \times 2^{..390625}\ 2^2 = 1.0724371 \times 1.3180796 \times 4 = 5.6542299$

$324^{.3} = 5.6645$ error $= 5.664 - 5.654 = .010$

324 multiplied	2 times	4 times	8 times	10 times
	X=324x324	Y=X x X	Z=Y x Y	X x Z
	104976	$1.1019961\ 10^{10}$	1.2143953×10^{20}	$1.2748236\ 10^{25} = 324^{10}$

$324^{3} = 5.66 \times 1.2748\ 10^{25} \quad = 7.22\ 10^{25}$

H.Summary

$a^3 = = 1 \times a \times a \times a = a \times a \times a$	Definition
$a^b a^c = a^{b+c}$	Add exponents
$a^b / a^c = a^{b-c}$	Subtract exponents
$a^0 = 1$	A number raise to 0 is 1
$a^1 = a$	One is the identity element
$a^{-b} = 1/a^b$	a negative exponent inverts
$(a^b)^c = a^{b \times c}$	Exponents of exponents are mult.
$(a^b c^d)^e = a^{be} c^{de}$	

$\log_b (a \times c) = \log_b(a) + \log_b(c)$
$\log_b (a / c) = \log_b(a) - \log_b(c)$
$\log_b (a^c) = c \log_b (a)$
$\log_b(1) = 0$
$\log_b (b) = 1$
$\log_{b2} (a) = \log_{b2} (b1) \times \log_{b1} (a)$
$\log_{b2}(b1) = 1/\log_{b1} (b2)$

This chapter is the essence of the book in trying to give the reader a true understanding of the meaning of exponents and logarithms in the framework of the operations of addition and multiplication.

I. Problems

1. $64^{1\ 5/6} =$
2. $\mathrm{Log}_e (2) =$
3. $\mathrm{Log}_e (3) =$
4. $\mathrm{Log}_e (12) =$
5. $3.14^{3.14} =$
6. Derive 7 logarithm formula
7. Derive 8 exponential formula
8. Find $10^{1/32}$
9. Find $10^{.1}$ $10^{.01}$ $10^{.001}$ $10^{.0001}$ $10^{.00001}$
10. Find $\log_{10}(2)$ using table from problem 8.
11. Check the result from problem 9.