Chapter II—Series

A. Infinite Series

We discovered that we could write the expression e^x as the sum of a series of terms:

$$e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + \ldots$$

The above expression is an *infinite series*. Let us look at one type of infinite series called a *geometric series*. Such a series is defined as one in which any term is found by multiplying the previous term by a constant value. The following is an example:

$$1 + 1/2 + 1/4 + 1/8 + \ldots$$

Each term is the preceding term multiplied by ½. If we wanted to calculate the sum of such a series we could do it as follows. Here "s" stands for the sum, "r" for the constant value multiplier, and "a" is the first term:

$$s = a + a(r) + a(r^2) + a(r^3) + \ldots + a(r^{(n-1)})$$
$$sr = a(r) + a(r^2) + a(r^3) + \ldots + a(r^{(n-1)}) + a(r^n) \qquad \text{Multiply each term by r.}$$

Subtracting the first from the second, we get:

$$\begin{aligned} s - sr &= a - ar^n \\ s(1 - r) &= a(1 - r^n) \\ s &= a(1 - r^n) / (1 - r) \end{aligned}$$

For a numerical example let us choose a = 1, r = ½, and n = 4. Then

$$\begin{aligned} s &= 1(1 - (\tfrac{1}{2})^4) / (1 - \tfrac{1}{2}) \\ &= 2(1 - (\tfrac{1}{2})^4) \\ &= 2 - (\tfrac{1}{2})^3 \\ &= 2 - \tfrac{1}{8} \\ &= 1\tfrac{7}{8}. \end{aligned}$$

As n becomes very, very large, if $r < 1$, r^n becomes very small (close to zero). Then

$$\begin{aligned} s &= a(1 + 0) / (1 - r) \\ &= a / (1 - r). \end{aligned}$$

In our example with a = 1 and r = ½, the sum s = 2. We say that s is the *limit* as $n \to \infty$.

Consider the repeating decimal .333⅓.

$$\begin{aligned} s &= .3 \times (1 - .1) \\ &= .3 / .9 \\ &= 3 / 9 \\ &= 1/3. \end{aligned}$$

Thus, we can find an answer to a fractional part in which the pattern repeats itself.

Some infinite series do not approach a limit and some others fluctuate between limits. Such series are called *divergent* series. The series we were dealing with, where r was less than 1, is called a *convergent* series. Consider the following series:

$$1/2 - 1/4 + 1/8 - 1/16 + \dots$$

In this series $a = ½$ and $r = -½$. Then

$s = a/(1-r) = (½)/(1+½) = (½)/(3/2) = ⅓$. We say that s converges to a limit of ⅓.

This is a way of using a bisecting technique to trisect something in the limit. An alternating sign series has an interesting property: if each term is smaller than the next, each succeeding term gives a number alternately greater than and less than the final answer and converges to it.

1/2	= .5
1/2 – 1/4	= .25
1/2 – 1/4 + 1/8	= .375
1/2 – 1/4 + 1/8 – 1/16	= .3125
1/2 – 1/4 + 1/8 – 1/16 +1/32	= .34375

The final limit is .33333333**...**

We are dealing with infinite series, so arithmetic series are not too useful, because they do not converge. Just for completeness let us look at one anyway.

$$\begin{aligned} s &= a_1 + a_2 + \dots + a_{n-1} + a_n \\ a_n &= a_1 + (n-1)b \\ a_m + a_{n+1-m} &= a_1 + (m-1)b + a_1 + (n-m)b = a_1 + a_1 + (n-1)b = a_1 + a_n \end{aligned}$$

$$\begin{aligned} s &= a_1 + a_2 + \dots + a_{n-1} + a_n \\ s &= a_n + a_{n-2} + \dots + a_2 + a_1 \\ 2s &= (a_1 + a_n) + (a_2 + a_{n-1}) + \dots + (a_{n-1} + a_2) + (a_n + a_1) \quad \text{adding the two equations} \\ &= n(a_1 + a_n) \\ s &= n(a_1 + a_n)/2 \end{aligned}$$

We can see that as n becomes very, very large, as long as a_1 and a_n are not zero, the sum becomes unlimited in size.

B. Standardized Form

We found that we can write any number in the form of a real part plus an imaginary part. We also discovered fractions and radicals. Let us look at some interesting examples.

Example 1: $1/2 \;/\; (1 + 1/2) = 1/2 \;/\; (3/2) = 1/2 \times 2/3 = 1/3 = .33333⅓$.

We changed the initial division into the ratio of two integers and then into a decimal.

Example 2: $1.3 / .4 = 13/4 = 3.25$.

We changed the above division into the ratio of two integers and then to a decimal.

Example 3: $1 / (1 + 2^{.5}) = (1 - 2^{.5})/((1-2^{.5}) * (1 + 2^{.5})) = (1 - 2^{.5}) / (1 - 2) = \mathbf{2^{.5} - 1} = .414$.

We eliminated the radical from the denominator and then changed the number to a decimal.

Example 4: $\mathbf{1 / (2 + i)}$

$$\begin{aligned} &= (2 - i) / ((2 - i)(2 + i)) \\ &= (2 - i) / (4 + 1) \\ &= \mathbf{2 / 5 - (1 / 5)i} \\ &= .4 - .2i. \end{aligned}$$

In this complex division we expressed the answer in terms of a real part plus an imaginary part.

Example 5: $\mathbf{e^{(3 + 2i)}} = (e^{3})(e^{2i}) = \mathbf{e^{3}(\cos(2) + i \sin(2))}$

Again we expressed this in the form of a real plus an imaginary part.

Example 6: $\ln (3 + 4i) = \ln (5 \text{ x } (3/5 + 4/5i)) = \ln (5 \text{ x } e^{(i \arcsin (4/5))}) = \ln 5 + i \arcsin(4/5)$

Here we also have the answer in the form of a real plus an imaginary part.

One reason to get numbers into a standard form is to make it easy to compare two or more numbers to see if they are equal.

To achieve the standard form we try to eliminate radicals, fractions, and complex numbers from the denominator. One "trick" is to take advantage of the fact that

$$(x + y)(x - y) = x^2 - y^2.$$

If $y = ai$ or $y = \sqrt{a}$ then $y^2 = -a$ or a, eliminating the i or the $\sqrt{\ }$.

C. Generating Functions

A generating function is an infinite power series in which we are interested in the coefficients of the exponential terms. The function $(1 + y)^n$ was one example. We would like to look at another series as an example that uses several of the mathematical techniques we have learned so far. The example we shall use is the Fibonacci Series:

0 1 1 2 3 5 8 13 21 34 55 **...**

where $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, $f_5 = 5$, and so forth.

The generating function is:

b	$= f_0 + f_1x + f_2x^2 + f_3x^3 + \dots$	
bx	$= f_0x + f_1x^2 + f_2x^3 + f_3x^4 + \dots$	Multiply by x.
$b(1 + x)$	$= f_0 + (f_0 + f_1)x + (f_1 + f_2)x^2 + (f_2 + f_3)x^3 + (f_3 + f_4)x^4 + \dots$	Add series together.
f_n	$= f_{n-1} + f_{n-2}$	Definition.
$b(1 + x)$	$= f_0 + (f_2)x + (f_3)x^2 + (f_4)x^3 + (f_5)x^4 + \dots$	Substitution.
$b(1 + x)x$	$= f_0x + (f_2)x^2 + (f_3)x^3 + (f_4)x^4 + (f_5)x^5 + \dots$	Multiply by x.
	$= f_0x + b - f_0 - f_1x$	Substitution.
b	$= (f_0(x - 1) - f_1x) / (x^2 + x - 1)$	
	$= -x / (x^2 + x - 1)$	Evaluate b.
$x^2 + x - 1$	$= 0$	
x	$= (-1 \pm \sqrt{5}) / 2$	Solve the quadratic.
$Ø$	$= (1 + \sqrt{5}) / 2$	Definition.
$1 / Ø$	$= 2 / (1 + \sqrt{5})$	Calculation.
	$= (\sqrt{5} - 1) / 2$	

$-x / (x^2 + x - 1) = a / (x - (-Ø)) + c / (x - 1 / Ø)$ — Partial fractions.

Multiply by $(x - (-Ø))(x - 1 / Ø)$ and equate coefficients x and the constant, and we get

$$a + c = -1 \quad \text{and}$$
$$-a / Ø + cØ = 0.$$

Solve the simultaneous equations:

a	$= -1 - c$	Solve for a.
$-a + Ø^2c$	$= 0$	
$1 + c + Ø^2c$	$= 0$	Substitute for a.
c	$= -1 / (1 + Ø^2)$	

Substitute $Ø = (1 + \sqrt{5}) / 2$

$$Ø^2 + 1 = (6 + 2\sqrt{5}) / 4 + 4 / 4$$
$$= (10 + 2\sqrt{5}) / 4$$
$$= (\sqrt{5})(\sqrt{5} + 1) / 2$$
$$= \sqrt{5}\, Ø$$

Evaluate a and c:

$$c = -1 / (\sqrt{5}\, Ø)$$
$$a = -1 - c$$
$$= -1 + 1 / (1 + Ø^2)$$
$$= -Ø^2 / (1 + Ø^2)$$
$$= -Ø^2 / \sqrt{5}\, Ø$$
$$= -Ø / \sqrt{5}$$

Substituting in the partial fraction:

$$
\begin{aligned}
b &= a / (x - (-Ø)) + c / (x - 1 / Ø) \\
&= (1 / \sqrt{5})(-Ø / (x + Ø) - (1 / Ø) / (x - 1 / Ø)) \\
&= (1 / \sqrt{5})(-1 / (1 + x / Ø) + 1 / (1 - Øx)) \\
&= (1 / \sqrt{5})(1 / (1 - Øx) - 1 / (1 + x / Ø)) \\
&= (1 / \sqrt{5})(1 / (1 - ((\sqrt{5} + 1) / 2)x) - 1 / (1 - ((1 - \sqrt{5}) /2)x))
\end{aligned}
$$

Dividing: $1 / (1 - dx) = 1 + dx + (dx)^2 + (dx)^3 + \ldots$

Expanding the partial fractions, we get:

$$b = \sum_{n=0}^{\infty}(1 / \sqrt{5}) (((\sqrt{5} + 1) / 2)^n - ((1 - \sqrt{5}) / 2)^n) x^n$$

Therefore,

$$f_n = (1 / \sqrt{5}) (((\sqrt{5} + 1) / 2)^n - ((1 - \sqrt{5}) / 2)^n)$$

This is a worthwhile example because it provides us practice in multiplying and adding radicals, using partial fractions, dividing by a polynomial, and solving simultaneous equations.

D. Practice

1. Find the rational value for .428571428571428**...**

2. Find the fourth term of the Fibonacci Series using the function defined in Section C of this chapter, Generating Functions.

3. Put into standardized (canonical) form:

   a) $(2 + 3i) / (\sqrt{2} + 4i)$

   b) $(-1 -\sqrt{3i})^3$

   c) $(26 + 15\sqrt{3})^{1/3}$

   d) $(2 + 3i)^{1/4}$