Chapter I—A Very Special Function

A. Introduction

The exponential function is the key that ties many of the topics of advanced mathematics together. Mathematics progresses from addition to multiplication and then to exponentiation. While the first two of those operations are commutative, exponentiation is not. With addition we discovered negative numbers; with multiplication, fractions; and with exponentials, complex numbers. With this knowledge we were able to define the general form of any number as

$$Ae^{2\pi ni + ix}$$

where A and x are any rational or irrational numbers and n is any integer from $-\infty$ to ∞. Because this information is so important, we are going to derive it.

Let us look at the expression $\mathbf{(1 + y)^n}$, where y is any number and n is any integer. We'll use three examples, where n is, in turn, 2, 3, and 4:

n=1:
$$\begin{array}{r} 1 + y \\ \underline{1 + y} \\ 1 + y \\ \underline{y + y^2} \\ 1 + 2y + y^2 \end{array}$$

n=2:
$$\begin{array}{r} 1 + 2y + y^2 \\ \underline{1 + y \qquad\;} \\ 1 + 2y + y^2 \\ \underline{y + 2y^2 + y^3} \\ 1 + 3y + 3y^2 + y^3 \end{array}$$

n=3:
$$\begin{array}{r} 1 + 3y + 3y^2 + y^3 \\ \underline{1 + y \qquad\qquad\quad} \\ 1 + 3y + 3y^2 + y^3 \\ \underline{y + 3y^2 + 3y^3 + y^4} \\ 1 + 4y + 6y^2 + 4y^3 + y^4 \end{array}$$

We see the pattern of Pascal's Triangle from the coefficients:

n			
1	1 1		1 1
2	1 2 1		1 2 1
3	1 3 3 1	or	1 3 3 1
4	1 4 6 4 1		1 4 6 4 1
5	1 5 10 10 5 1		1 5 10 10 5 1

We use the notation P(n) to mean the number of *permutations* of n things—the number of ways we can arrange or sequence those things. That number is n! (read as "n factorial"). Suppose, though, we want only a subset of m things from that set of n things. We denote the number of possible such subsets—the number of *combinations* of n things taken m at a time—as C(m,n). That number is n! / [(m!)(n – m)!]. The order of things within each combination is immaterial. If the order did matter, we would have the number of permutations of n things taken m at a time, which we would denote as P(m,n), and which would equal

$$n(n-1)(n-2)\ldots(n-m+1)$$

or n! / (n – m)!.

Let us look at the values of P(m,n) when n = 4 and m is, respectively, 1 or 2 or 3:

$$P(1,4) = 4! / (3!) = 4; \quad P(2,4) = 4! / (2!) = 12; \quad \text{and} \quad P(3,4) = 4! / (1!) = 24.$$

If 0! = 1 then $P(0,4) = 4! / (4!) = 1$ and $P(4,4) = 4! / (0!) = 24$.

So, the values of P(m,4) as m goes from 0 to 4 are, respectively, 1, 4, 12, 4, and 24.

B. Symmetry of the Combinatorials

At first it might seem as though someone with great insight had accidentally found a function that predicted or explained the coefficients in the binomial expansion. If we look at each coefficient of y^m we could use the selection of m things from n to explain it. Another possibility could be that because each term is formed by adding the coefficient directly above it to the one to the left of the one above it, then only the function C(m,n) satisfies this constraint.

Let us now look at a definition of C(n) that can explain why 0! = 1, and later why C(m,n) = 0 if we were to imagine a case where m was greater than n.

First, let us define $1 / C(n-1) = n / C(n)$, where $C(1) = 1$.

Then, for n = 1, $1 / C(0) = 1 / C(1) = 1 / 1 = 1$.

Therefore C(0) = 1.

We notice that there is a symmetry between C[(n – m),n] and C(m,n):

$$C[(n-m),n] = n! / ((n-m)!(n-(n-m))!) = n! / (((n-m)!)(m!)) = C(m,n).$$

Let us quickly check that this formula is valid for the examples we've looked at.

$$C(0,2) = C(2,2) = 2! / (0!)(2!) = 1$$
$$C(1,2) = 2! / (1!)(1!) = 2$$
$$C(0,3) = C(3,3) = 3! / (0!)(3!) = 1$$
$$C(1,3) = C(2,3) = 3! / (1!)(2!) = 3$$
$$C(0,4) = C(4,4) = 4! / (0!)(4!) = 1$$
$$C(1,4) = C(3,4) = 4! / (1!)(3!) = 4$$
$$C(2,4) = 4! / (2!)(2!) = 6$$

We can generalize the binomial expansion by using a summation series:

$$(1 + y)^n = \sum_{m=0}^{n} C(m,n)(y^m)$$

As we become familiar with a mathematical shorthand we are able to grasp concepts expressed in that notation much more easily.

C. Range of the Combinatorials

With our definition of C(n), we found that $0! = C(0) = 1$. What happens mathematically when n is less than 0, that is, when we have a negative number of things to group into combinations?

$$1 / C(-1) = 0 / C(0) = 0.$$
$$1 / C(-2) = -1 / C(-1) = (-1)(0) = 0.$$
$$1 / C(-3) = -2 / C(-2) = (-2)(-1 / C(-1)) = (2!)(0) = 0.$$

We can then see that

$$C(n + 1,n) = C(n) / [(C(n + 1)][C(n - (n + 1))] = C(n) / ([C(n + 1)][C(-1)]) = 0.$$

Therefore, if $m > n$, then $C(m,n) = 0$.

Because $C(m,n) = C(n) / (C(n - m) C(m))$, for $m = -m$, we get

$$\begin{aligned} C(-m,n) &= C(n) / C(n + m) C(-m) \\ &= (C(n) / C(n + m))[1 / C(-m)] \\ &= (C(n) / C(n + m))(0) \\ &= 0. \end{aligned}$$

Therefore, if $m < 0$, then $C(m,n) = 0$, and from above, for $m > n$, $C(m,n) = 0$.

D. Pascal's Triangle

In Pascal's triangle we observed that each element is formed by the sum of certain elements above it. We would like to show that our definition of C(m,n) follows this rule.

$$\begin{aligned} \mathbf{C(m,n) + C(m + 1,n)} &= n! / (m! (n - m)!) + n! / ((m + 1)! (n - m-1)!) \\ &= (n! / ((m + 1)!(n - m)!)) \times (m + 1 + n - m) \\ &= (n + 1)! / ((m + 1)! (n + 1 - m - 1)!) \\ &= \mathbf{C(m + 1,n + 1)} \end{aligned}$$

At the right we have reformatted the Pascal Triangle as a right triangle.

n/m	0	1	2	3	4	5
0	1	0	0	0	0	0
1	1	1	0	0	0	0
2	1	2	1	0	0	0
3	1	3	3	1	0	0
4	1	4	6	4	1	0
5	1	5	10	10	5	1

E. Sum of the Combinatorials

By looking at the combinatorials with different mathematical tools we can find some relationships that are interesting. Consider the following:

$$(1 + y)^n = \sum_{m=0}^{n} C(m,n)(y^m)$$

For example, if $y = 1$, then $\sum_{m=0}^{n} C(m,n) = 2^n$

$$\sum_{p=0}^{2n} C(p,2n) = \sum_{p=0}^{n} (C(2p,2n) + C(2p+1,2n)) = 2^{2n}$$

Because $C(2n+1,2n) = 0$ (because $2n+1 > 2n$),
and $C(2p,2n) = C(2p-1,2n-1) + C(2p,2n-1)$ (sum of terms above),
we have

$$\sum_{p=0}^{n} C(2p,2n) = \sum_{p=0}^{n} C(2p-1,2n-1) + C(2p,2n-1) = \sum_{p=0}^{2n} C(p,2n-1) = 2^{(2n-1)}.$$

And because $C(-1,2n-1) = C(2n,2n-1) = 0$ for $n > 0$, we have

$$\sum_{p=0}^{n} C(2p+1,2n) = \sum_{p=0}^{2n} C(p,2n) - \sum_{p=0}^{n} C(2p,2n) = 2^{2n} - 2^{(2n-1)} = 2^{(2n-1)} \text{ x } (2-1) = 2^{(2n-1)}.$$

odd terms = all terms - even terms

Then, for odd terms (with $n > 0$)

$$\mathbf{(1 / (2n)!) \text{ x } \sum_{p=0}^{n} C(2p+1,2n) = 2^{(2n-1)} / (2n!) = \sum_{m=0}^{n-1} 1 / ((2n-2m-1)!(2m+1)!)}$$

and for even terms (again, with $n > 0$)

$$\mathbf{(1 / (2n)!) \text{ x } \sum_{p=0}^{n} C(2p,2n) = 2^{(2n-1)} / (2n!) = \sum_{m=0}^{n} 1 / ((2n-2m)!(2m)!)}$$

Consider the cases for $n = 3$:

$$1 / (6!)(0!) + 1 / (4!)(2!) + 1 / (2!)(4!) + 1 / (0!)(6!) = (½)(2^6) / 6! = 2^5 / 6! = 32 / 720.$$

$$1 / (5!)(1!) + 1 / (3!)(3!) + 1 / (1!)(5!) = (½)(2^6) / 6! = 2^5 / 6! = 32 / 720.$$

Let us see if it is true:

$$1 / (6!)(0!) + 1 / (4!)(2!) + 1 / (2!)(4!) + 1 / (0!)(6!) = 1 / 720 + 1 / 48 + 1 / 48 + 1 / 720 = (1 + 15 + 15 + 1) / 720 = 32 / 720$$

$$1 / (5!)(1!) + 1 / (3!)(3!) + 1 / (1!)(5!) = 1 / 120 + 1 / 36 + 1 / 120 = (6 + 20 + 6) / 720 = 32 / 720$$

One of the great satisfactions in mathematics occurs when an algebraic derivation is validated by the substitution of numerical values. At this stage, this formulation is a mere mathematical curiosity, but later we will see why this formulation for summing up the alternate coefficients of the binomial expansion is so interesting.

F. Determining the Transcendental Number "e"

Let us go back to the binomial expansion.

$$(1 + a)^b = 1 + (b / 1!)(a) + [b(b - 1) / 2!](a^2) + [b(b - 1)(b - 2) / 3!](a^3) + \ldots$$

Let us substitute some new variables for the values of a and b. Let $a = 1 / n$ and $b = nx$.

Then $((1 + 1/ n)^{nx}) = ((1 + 1/n)^n)^x$

$$(1 + a)^b = 1 + nx / n + ((nx)(nx - 1)) / ((n^2)(2!)) + ((nx)(nx - 1)(nx - 2)) / ((n^3)(3!)) + \ldots$$
$$= 1 + x + (x(x - 1/n)) / 2! + (x(x - 1/n)(x - 2/n)) / 3! + \ldots$$

As n gets very, very large (we say "as $n \to \infty$"):

$$e = (1 + 1 / n)^n = (1 + 1 / \infty)^\infty$$

$$e^x = (1 + a)^b = 1 + x + x^2 / 2! + x^3 / 3! + \ldots$$

We can determine properties of the cosine and sine functions by substituting iy for x:

$$e^{(iy)} = (1 - y^2 / 2! + y^4 / 4! + \ldots) + i(y - y^3 / 3! + y^5 / 5! - \ldots)$$
$$= c(y) + i\ s(y), \qquad \text{where } i^2 = -1.$$

We can rewrite e(y), s(y), and c(y) as follows:

$$e(y) = \sum_{m=0}^{\infty} (y^m) / C(m)$$

$$s(y) = \sum_{m=0}^{\infty} (-1)^m (y^{(2m+1)}) / C(2m+1) \qquad s(-y) = -s(y)$$

$$c(y) = \sum_{m=0}^{\infty} (-1)^m (y^{(2m)}) / C(2m) \qquad c(-y) = c(y)$$

Now let us look at [s(y)][s(y)] as follows:

$$[s(y)][s(y)] = \sum_{p=1}^{\infty} (-1)^p \sum_{m=0}^{p} \big(1 / ([C(2p-2m-1)][C(2m+1)])\big)y^{2p}$$

You actually have to expand the first few terms of the product before you can see the pattern that leads to the above formula. We recognize the sum of the reciprocal factorial products from the previous section and make the appropriate substitutions:

$$= \sum_{p=1}^{\infty} (-1)^{(p+1)} [2^{(2p-1)}/(2p)!][y^{2p}] = \sum_{p=1}^{\infty}(-1)^{p+1}(½)(2y)^{2p} / (2p)!$$

Now let us look at [c(y)][c(y)]:

$$[c(y)][c(y)] = 1 + \sum_{p=1}^{\infty}(-1)^p \sum_{m=0}^{p} \big(1 / (C(2p-2m)[C(2m)])\big)y^{2p}$$

$$= 1 + \sum_{p=1}^{\infty}(-1)^p [2^{(2p-1)} / (2p)!][y^{2p}]$$

$$= 1 + \sum_{p=1}^{\infty} (-1)^p [½](2y)^{2p}/ 2p!$$

Looking at these two products [s(y)][s(y)] and [c(y)][c(y)] we see the following relationships:

$$[c(y)][c(y)] = ½ + c(2y) / 2$$
$$c(2y) = 2[c(y)][c(y)] - 1$$

$$[c(y)][c(y)] + [s(y)][s(y)] = 1$$

From trigonometry we will learn that $\cos(2y) = 2\cos^2(y) - 1$ and that $\cos^2(y) + \sin^2(y) = 1$.

Thus, c(y) has the same properties as cos(y), and s(y) has the same properties as sin(y). We derived these properties from algebraic operations on the power series. Now let us work with the functions themselves:

$$e^{ia} \times e^{ib} = e^{i(a+b)}$$
$$= \cos(a + b) + i(\sin(a + b))$$
$$= (\cos(a) + i(\sin(a)))(\cos(b) + i(\sin(b)))$$

$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ Equating the real parts.
$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$ Equating the coefficients of i.

Remembering that $\sin(-a) = -\sin(a)$ and $\cos(-a) = \cos(a)$, we have

$$\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$
$$\sin(a - b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$

Let us look at the following:

$$(e^{ia})(e^{-ia}) = e^0 = 1$$
$$= [\cos(a) + i(\sin(a))][\cos(a) + i(\sin(a))]$$
$$= \cos^2(a) + \sin^2(a)$$

Because $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, if we substitute b = a, we get:
$$\cos(2a) = \cos^2(a) - \sin^2(a)$$
$$= 2\cos^2(a) - 1$$

These are the same formulae that we got from using the power series. This reconfirms our confidence in the consistency of the formulations.

Could these functions be periodic or modular in that they have the same values as "a" gets bigger and bigger? We know from the formula that

$$\sin(0) = 0 \text{ and } \cos(0) = 1.$$

Since $\cos^2(a) + \sin^2(a) = 1$, there must be some value for "a" such that $\cos(p) = 0$ and $\sin(p) = 1$.

Because $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$
and
$$\sin((a - b) = \sin(a)\cos(b) - \cos(a)\sin(b),$$

if we substitute p for a, we get

$$\cos(p - b) = \cos(p)\cos(b) + \sin(p)\sin(b) = (0)\cos(b) + (1)\sin(b) = \sin(b)$$
and
$$\sin(p - b) = \sin(p)\cos(b) - \cos(p)\sin(b) = (1)\cos(b) - (0)\sin(b) = \cos(b).$$

Since cos(2b) = 2cos(b)cos(b) - 1
cos(2p) = -1 and sin(2p) = 0
cos(4p) = 2cos(2p)cos(2p) - 1 = 1 and sin(4p) = 0
cos(3p) = cos(2p)cos(p) - sin(2p)sin(p) = (-1)(0) – (0)(1) = 0
sin(3p) = sin(2p)cos(p) + cos(2p)sin(p) = (0)(1) + (-1)(1) = -1.

Let us make a table:

a	sin a	cos a
0	0	1
p	1	0
2p	0	-1
3p	-1	0
4p	0	1

Thus, the sine and cosine functions have periods of 4p and half periods of 2p = π. These would be roots of a large polynomial, so we could write the sine series as a product series. We used a square term because the sine is periodically symmetrical about 0. That is, sin(-x) = -sin(x).

$$s(y) = y\prod_{n=0}^{\infty} (1- (y / (n\pi))^2)$$

$$= y - (y^3)(\pi^{-2})(1 + 1/4 + 1/9 + 1/16 + \ldots) + \ldots$$

$$= y - y^3/ 3! + \ldots$$

Equating the coefficients of y^3, we get $\pi^2 = (6)(1 + 1/4 + 1/9 + \ldots)$. Though this will give us an estimate of pi (π), it will take many calculations to obtain even a few significant digits.

Starting with the binomial expansion, we were able to discover e and π, two irrational or in these cases, transcendental numbers. Neither can be represented as the ratio of two integers. Mathematicians do not proceed in such a well-directed approach to discovery. Usually there are lots of false starts taken before we find a fruitful path. Much is based on trying to create and find patterns. We try looking at problems in different ways to insure mathematical consistency.

Because of the periodic nature of e^{iy}, we find that we can write any number as:

$$Ae^{2\pi ni + ix} = \mathbf{e^{a+2\pi ni + ix}}, \text{ where } e^a = A.$$

G. Numerical Evaluation of e

Learning how to determine the values of irrational numbers is important so that you can develop a better number sense and can practice mathematical manipulation and organization.

$$e^y = 1 + y + y^2 / 2! + y^3 / 3! + y^4 / 4! + \ldots$$

Then for $y = 1$, $e^1 = e = 1 + 1 + 1/2 + 1/6 + 1/24 +$ **...**

Each number is obtained by dividing the previous number by the next counting number, so we can calculate "e" by successive approximations as follows:

e =	1.000000000	
	+1.000000000	
	+0.500000000	previous line divided by 2
	+0.166666666	previous line divided by 3
	+0.041666666	previous line divided by 4
	+0.008333333	previous line divided by 5
	+0.001388888	etc.
	+0.000198412	
	+0.000024801	
	+0.000002755	
	+0.000000275	
	2.718281726	

With "e" we are fortunate that the series converges rapidly so that we do not have to do many calculations. Let us determine $e^{.5}$:

$$e^{.5} = 1 + 1 / 2 + 1 / (4 \times 2!) + 1 / (8 \times 3!) + \ldots$$
$$= 1.64872170687$$

Then $e^{.5} \times e^{.5} = e^1 = e = 1.6487217 \times 1.6487217 =$ **2.718281...**

The rules for exponentiation hold for the values calculated from this series.

H. Practice

1. Calculate:

   a. 5!

   b. 6! / (4!)(2!)

   c. (-6!) / ((-4!)(21!))

2. Calculate: $(a + b)^6$

3. Find the Pascal coefficients for $(a + b)^7$.

4. Calculate e to the tenth decimal place.

5. Find the sum of the 11th row of the following:

1) 3 5
2) 3 8 5
3) 3 11 13 5
4) …
…
11) __ __ __ __ __ __ __ __ __ __ __ __ Sum = ________