VIII. Negative Numbers, Discovery. (3rd to 4th)

A. Introduction

Because of the way we are usually introduced to negative numbers, over time they should puzzle us. Negative numbers are a notation that begin to complete our set of numbers and simplify calculations. We discovered 0 by counting backwards. Suppose that we continued counting backwards, what should we name these positions on the number line. We could use letters of the alphabet. One time I asked a first grader to name them. He named them as follows:

$$-4 -3 -2 -1 -0 \ 0 \ 1 \ 2 \ 3 \ 4$$

I knew better about the –0, but how was I going to explain that the –0 occupied the same position on the number line as zero. For those of you who believe that zero is nothing, this does not seem a problem, but for me it did.

What we will come to learn is that negative numbers result from notation and we try to illustrate how this came about.

B. Definition

Let us define a negative number such that $n+-n=0$. Where $-n$ is the negative of n. For the time being let us consider $-n$ just a naming protocol. Let us consider the number before 0. If would be $A+1=0$ and the number before A would be $B+1=A$. Now let us do some subtraction:

$$A+1=0$$
$$A \ \ =0-1$$

$$B+1=A$$
$$A+1=(B+1)+1=B+2=0$$
$$B=0-2$$

For notation purposes, let us drop the zero and we have:

$$A= 0-1 = -1$$
$$B= 0-2 = -2$$

The subtraction symbol has become a negative sign. For convenience, we use the same symbol though some mathematicians may elevate it to distinguish it from the subtraction symbol. In a diagram we summarize what we have done:

B	A	0	1	2
0-2	0-1	0	1	2
-2	-1	0	1	2

That is nice but what happen to –0?

$$0+0=0$$
$$0 \ \ =0-0 = -0 \ \text{ by notation}$$

Aha, -0 is equal to zero. We have to be careful of an assumption; is $0+0=0$? We know: $1+0=1$.

$$1+1=2$$
$$1+1+0=2$$
$$2+0=2$$

Since $1+1=1+0+1+0= 2+0+0= 2= 2+0$

Then $0+0=0$

C. The Rules

We have shown how the definition n+-n=0 leads to the notational definition of a negative number. The first rule we derive as follows

$$n+-n=0$$
$$n = 0 - -n \quad \textbf{rule 1a}$$

When we subtract a negative number, we make the number positive and add the number. Related to this rule, we have:

$$-n = 0 - n \quad \textbf{rule 1b}$$

Adding a negative number is the same as subtracting a positive number

For the next rule we have:

$$n+-n + m + -m = 0 + 0 =0$$
$$(n+m) + (-n+-m) = 0$$
$$-n+ -m = 0 - (n+m) = -(n+m) \quad \textbf{rule 2}$$

When adding two negative numbers, add them as if they were positive numbers and then make the sign negative.

For rule number three, when we subtract two numbers. subtract the smaller from the larger and use the sign of the larger number:

$$5 - 3 = 2+3-3= 2+ 0 = 2 \qquad \textbf{rule 3a}$$
$$3 - 5 = 3+ (-3 +-2) = 0 + -2 =-2 \quad \textbf{rule 3b}$$

D. Testing for Understanding

1.	$5+3 = 8$	
2.	$5+-3 = 5-3 = 2$	rule 1b
3.	$-5+3 = 3-5 = -(5-3) = -2$	rule 1b rule 3b
4.	$-5+ -3 = -(5+3) =-8$	rule 2
5.	$5-3 = 2$	
6.	$5 - -3 = 5 +3 - 8$	rule 1a
7.	$-5 - 3 = -5 + -3 = -(5+3)=8$	rule 1b
8.	$-5 - -3 = -5+3 = 3-5=-(5-3) =-2$	rule 1a rule 1b rule 3b

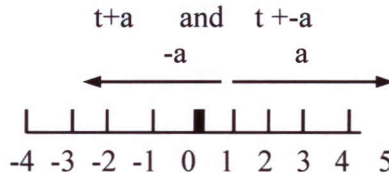**The general strategy is to change – - n to + n, and - n to + -n.**

E. Applications and manipulatives

The number line is another way of understanding the rules:

```
          <────────────────  -
          ────────────────>  +
  └─┬─┬─┬─┬─■─┬─┬─┬─┬─┘
  -4 -3 -2 -1  0  1  2  3  4
```

When subtracting two numbers such as h − t , put the head of the arrow at h and the tail at t. If the arrow points right, the answer is positive; if it points left, the answer is negative. The length of the arrow is the answer.

When adding two numbers place, the tail at the t and the length of the arrow is the magnitude of the number "a" and the arrow points to the right if the number is positive and to the left if "a" is negative. Where the head ends is the answer.

$$t+a \quad \text{and} \quad t +\text{-}a$$

```
              -a              a
          <────────    ──────────>
  └─┬─┬─┬─┬─■─┬─┬─┬─┬─┘
  -4 -3 -2 -1  0  1  2  3  4  5
```

Another way to look at this is to let copper pennies be positive 1 and blue pennies be negative 1. A copper penny plus a blue penny gives us zero. Now let us try to prove some of our negative number theorems.

$$\bullet + \bullet = 0$$

Let us prove that to subtract a negative number is the same as to add a positive number:

Add zero to the above since there are no blue pennies to take away:

Now take away the two blue pennies:

We now proved what we had set out to do.

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What is especially important in what we did was using the concept of adding zero to an equation to allow us to solve it. This is a technique that will carry on into high school algebra.

F. Measuring

It almost seemed that the negative numbers allow us to correct a problem in that we left out some numbers. A thermometer seems to illustrate this. When Fahrenheit was inventing the thermometer, he thought that the coldest substance was frozen salt water. He set that as his zero on the thermometer. When it was discovered that here were colder things, negative numbers were place on the thermometer. The coldest temperature is believed to be about –459 degrees Fahrenheit (-273 degrees Celcius). Buildings in the United States start at floor 1 whereas in England, they start a the ground floor (floor 0). Who would ever think that someone would build underground? In the United States parking garages are. Instead of using negative numbers they use P numbers. Here are examples of the two application of negative numbers:

```
20              2               2
10              1               1
 0             P1      or      P3
-10            P2              P2
-20            P3              P1
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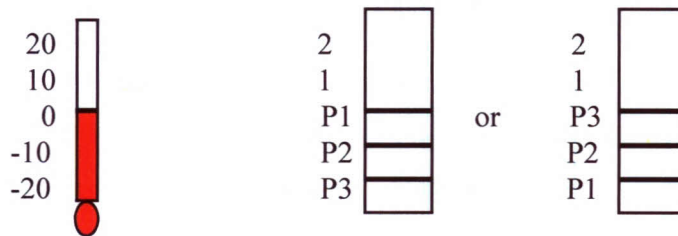
We can see that there are other ways we could have look at negative numbers when we consider some applications such as naming floors in buildings.

G. Multiplying negative numbers

To multiply negative numbers, we start off we the definition of multiplication:

$$3x -5 = =5 + -5 + -5 = -(5+5+5)= -(3x5)$$

To prove that the multiplication is commutative we use the distributive rule:

$$(3+-3)x5 = 0 \text{ x } 5 = 0$$
$$3x5 + -3x5 = 0$$
$$-3x5 = 0 -(3x5) = -(3x5)$$

This proof assumed that multiplication by zero was commutative.

$$(0 + 1) \text{ x } 5 = 1x5 = 5$$
$$0x5 + 1x5 = 0x5 + 5 = 5 = 0 + 5$$
$$\text{then } 0x5 = 0$$

This is true if 0 is a number that obeys the distributive rule. Eventually, in mathematics we get driven into a corner and have to make up new concepts or accept what seems reasonable.

Now here is where things become amazing:

$$(3+-3)x-5 = 0 \text{ x } -5 = 0$$
$$3x-5 + -3x-5 = -(3x5) + -3x-5 = 0$$
$$-3x-5 = 0 - -(3x5)= 3x5.$$

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A negative number times a negative number is a positive number. This seems hard to grasp. If we had defined multiplication as a count with negative numbers, we would have:

$$-3 \times 5 = 5 + 5 + 5 = 15$$

With this definition; $-3 \times -5 = -5 + -5 + -5 = -15$ where the product of two negative numbers is a negative number. Following through our distributive proof, then the product of two positive numbers would be a negative number. What a topsy turvy world in which we would live. Our mathematics is adjusted to match what we observe.

G. Summarizing

We define negative numbers by the following equation: $n + -n = 0$.
This lead to three major rules:

> Rule 1: When subtracting a negative, make it positive and add
> > Adding a negative number is the same as subtracting a positive number
> Rule 2: When adding two negative numbers add as positive numbers and make sign negative
> Rule 3: When subtracting two numbers, subtract the smaller from the larger and use the sign of the larger.

Proof of rule 1 is as follows:
$$n + -n = 0$$
$$n \quad = 0 - -n \quad \text{rule 1a}$$
$$-n = 0 - n \quad \text{rule 1b}$$

The proof of rule 2 is:
$$n + -n + m + -m = 0 + 0 = 0$$
$$-n + -m = 0 - (n+m)$$

We then show the application of negative numbers in measuring devices. Finally we showed how to multiply negative numbers:
$$-n \times m = -(n \times m)$$
$$m \times -n = -(n \times m)$$
$$-n \times -m = n \times m$$

To prove the last rule we used the distributive rule as follows:
$$(n + -n) \times -m = 0 \times -m = 0$$
$$-n \times m + -n \times -m = 0$$
$$-n \times -m = 0 - -(n \times m) = n \times m$$

Notice in our summary, we did not cover every topic. This is another technique in note taking if we want to insure that we remember the salient points. To much information overwhelms us and we forget. See the appendix for quiz examples.