Chapter VII—Negative Numbers and Fractions

A. Negative Numbers

Negative numbers and fractions have so many parallels that it is worth teaching them together. The student can then apply the patterns for learning negative numbers to that of fractions. Negative numbers and fractions extend the structure of numbers to a new dimension. If we were to write the numbers in a linear sequence we can see how the next number is found merely by adding one to the last number we wrote:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$$

One could then ask what happens when we extend the numbers to the left. First we would find the zero and then negative one followed by the rest of the negative numbers:

$$\ldots, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$$

To simplify this discovery, we think of the number one as the generator of our positive numbers, where for example 3 is a shorthand notation for $1 + 1 + 1$. We create a number called -1 which is an *annihilator* so that $-1 + 1 = 0$. This allows us to count backward. For example,

$$3 + -1 = 1 + 1 + 1 - 1 = 1 + 1 + 0 = 2.$$

When we add the -1 to 0 we get a standalone -1. We then define the other negative numbers in the same way we define -2: $-2 = -1 + -1$. Unfortunately, this approach fails to explain zero in proper perspective. However, it does show why there is not a -0 between -1 and 0.

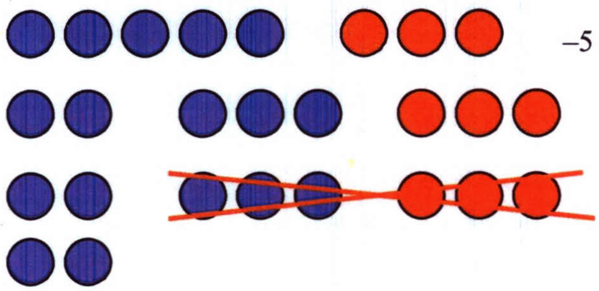
The key area to focus on is the definition of -1: $-1 + 1 = 0$. Once we appreciate this concept we can develop the rules for adding and subtracting negative and positive numbers.

B. Adding and Subtracting

Subtraction is defined as the inverse of addition, but rather than develop the rules for subtraction from the definitions, we are going to use a manipulative approach. The creative mind obeys the rules but also makes up rules and questions existing ones. We will use blue objects to represent negative numbers and red objects to represent positive numbers, with the rule being that one red object plus one blue object annihilate one another, leaving nothing (zero). To add all red or all blue objects we follow the traditional rules for addition or counting but use the sign of the color we are adding. For example two blues plus three blues is five blues:

$$-2 + -3 = -5.$$

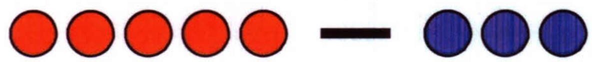
Adding a mixed set requires a different strategy. We match up the number of the objects with the maximum quantity to that of the one of the least quantity. Since there are equal numbers and each blue annihilates one red, we will be left with the color of the quantity which was greater. In order to get a match, we had to use subtraction to find out what was left over from the match. Let us follow the example:



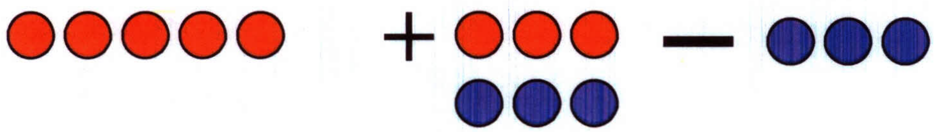
$$-5 + 3 = -2 + -3 + 3$$
$$= -2 + (-3 + 3)$$
$$= -2 + 0$$
$$= -2.$$

We got the –2 by subtracting 3 from 5 and then using the minus sign because there were more blue objects than red.

With subtraction we have to be creative. We would like to subtract –3 from 5. Let us follow the steps:



Since I do not have any blues that I can take away from, let me add zero to the 5 as follows:



When I take away the –3 (three blue objects), I am left with 8 red objects.

Thus, $5 - -3 = 8$. A rule to express our observation would be that when we subtract negative numbers (blue objects), we change the sign (color) and add. An analogous rule holds for subtracting positive numbers. This concept of adding zero without changing the values is a step used in algebra problems, for example, when we solve a quadratic equation.

So, with the simple rule that $-1 + 1 = 0$, we can develop the rules for subtraction:

1. $2 + -2 = 0$	Definition of a negative number.
2. $2 = 0 - -2$	Change the sign when subtracting and then add.
3. $-2 = 0 - 2$	Parallel proof.
4. $5 - -3 = 5 + 0 - -3$	
$= 5 + (3 + -3) - -3$	
$= 5 + 3 + (-3 - -3)$	
$= 5 + 3 + 0$	
$= 8 + 0$	
$= 8$	Added zero to solve a problem algebraically.

C. Multiplication of Negative Numbers

We are going to develop the rules for multiplying negative numbers solely from pattern recognition. We know that by the definition of multiplication: $-8 = -2 + -2 + -2 + -2 = 4 \times -2$ Look at the following pattern:

$$3 \times -2 = -6$$
$$2 \times -2 = -4$$
$$1 \times -2 = -2$$

On the left side we are decreasing the multiplier by one and on the right side we are subtracting –2 (adding 2) from the previous result. Let us continue the pattern:

$$3 \times -2 = -6$$
$$2 \times -2 = -4$$
$$1 \times -2 = -2$$
$$0 \times -2 = 0$$
$$-1 \times -2 = 2$$
$$-2 \times -2 = 4$$
$$-3 \times -2 = 6$$

Thus, from pattern recognition we observe that a negative number times a positive number is a negative number.

Let us try one more pattern:

$$3 \times 2 = 6$$
$$2 \times 2 = 4$$
$$1 \times 2 = 2$$
$$0 \times 2 = 0$$
$$-1 \times 2 = -2$$
$$-2 \times 2 = -4$$
$$-3 \times 2 = -6$$

We decreased the multiplier on the left side by one, and we subtracted 2 from the previous result. In this pattern we see that if we multiply a positive number and a negative number together the result is a negative number and that when we multiply two negative numbers together the result is a positive number. All these rules are discovered from being creative with patterns.

D. Defining Fractions

Fractions bring together all the math work that we do in elementary school. They complete another piece of the number structure picture as did negative numbers. With negative numbers we looked at the other end of the number line, while for fractions we look between the numbers. If we take a section of the number line between two numbers and break it into ten equal pieces, any one of those pieces would be called a tenth, because it would take ten of them to make one (a whole). I invented a notation for fractions to parallel that of negative numbers, where I call that tenth: **/10**.

Using the definition of a fraction, then:

$$/10 + /10 + /10 + /10 + /10 + /10 + /10 + /10 + /10 + /10 = 1.$$

We have defined a shorthand notation for the above. It's called multiplication. Thus, we have:

$$10 \times /10 = 1.$$

For subtraction we have $10 + -10 = 0$. The number one has a special property in multiplication, so we call it the *unary factor for multiplication*. The zero has a special property in addition, so we call it the *unary factor for addition*. These special properties are:

$$1 \times 10 = 10 \qquad \text{and} \qquad 0 + 10 = 10.$$

In other words, the result of multiplying any number by one, or adding zero to any number, is that same "any number," unchanged.

We define subtraction as the *inverse* operation of addition:

$$\begin{array}{lll} \text{If} & 5 + 3 & = 8 \\ \text{then} & 5 & = 8 - 3 \\ \text{or} & 3 & = 8 - 5. \end{array}$$

We define division as the inverse operation of multiplication:

$$\begin{array}{lll} \text{If} & 5 \times 3 & = 15 \\ \text{then} & 5 & = 15 / 3 \\ \text{or} & 3 & = 15 / 5. \end{array}$$

Because we use the subtraction sign as a minus sign, I decided to follow this pattern and use the division sign as an indicator of a fraction. If $10 \times /10 = 1$, then by the definition of division:

$$\begin{array}{ll} 10 & = 1 / /10 \\ /10 & = 1 / 10. \end{array}$$

Thus, 10 is the reciprocal of one tenth and one tenth is defined as 1 divided by ten. This matches what we already know about fractions, just using a different notation. Look at the parallel with negative numbers:

$$\begin{array}{llllll} 10 \times /10 & = 1 & \quad & 10 + -10 & = 0 & \\ 10 & = 1 / /10 & & 10 & = 0 - -10 & (1) \\ /10 & = 1 / 10 & & -10 & = 0 - 10 & (2) \end{array}$$

In statement (1) we see that the inverse of an inverse gives us our original number, and in (2) we define our new notation by an inverse operation. **Understanding this application of the pattern will help nurture our ability to make our own creative discoveries.** Whether or not someone else has made a discovery should be irrelevant if we made it on our own without prior knowledge. We are then in the same place in time as the person who discovered it originally.

In the above cases we have used numerical examples to illustrate the definitions. In algebra we would use symbols. For example, instead of using 10 as we did above, we could use "a." Then:

$$\begin{aligned} a \times /a &= 1 \\ a &= 1 / /a \\ /a &= 1 / a \end{aligned}$$

$$\begin{aligned} a + -a &= 0 \\ a &= 0 - -a \\ -a &= 0 - a \end{aligned}$$

E. Adding, Multiplying, and Dividing

When we define a new number concept, we test it under our known operations. We know that:

$$a + 2a + 3a = 6a.$$

From the Distributive Rule, if $a = /10$ then,

$$1 \times /10 + 2 \times /10 + 3 \times /10 = 6 \times /10$$

Let us check something out:

$$10 \times /10 = 1$$

7	$= 7$	
7×1	$= 7$	
$7 \times 10 \times /10$	$= 7$	
$10 \times (7 \times /10)$	$= 7$	Commutative Rule.
$7 \times /10$	$= 7/10$	Definition of division.

Going back to our previous problem, we then have:

$$1 \times /10 + 2 \times /10 + 3 \times /10 = 6 \times /10 = 6 / 10.$$

Rewriting, we have:

$$1 / 10 + 2 / 10 + 3 / 10 = 6 / 10$$

What we call a fraction is really the division of two numbers. People who have been taught in the conventional process will initially find it hard to grab this concept, because the conventional process is so ingrained in them and it requires people to admit that they have not be taught in a more effective fashion. One of the rules of science is that if you discover something new, you have to explain how it is consistent with what you already know. Thus, every operation that you did with fractions still works; it's just how you understand it that is different.

To multiply fractions we make use of the fact that multiplying anything by 1 gives us our original number.

3 x /3 x 2 x /2	= 1 x 1 = 1	
3 x 2 x /3 x /2	= 1	Commutative Rule.
(3 x 2) x (/3 x /2)	= 1	Association.
/3 x /2	= 1 / (2 x 3)	Definition of division.
/3 x /2	= /(2 x 3)	Definition of a fraction.

We could think of this in terms of manipulatives: Divide a load of bread into two parts (halving it). Divide each of these two parts into thirds. If we count all the parts, we have six. Thus,

1/2 x 1/3 = 1/6. See the picture:

/6 = 1/6

Dividing fractions is straight forward:

/3 / /2 = /3 x 1 / /2		
= /3 x (1 / /2)		
= /3 x 2		Inverse of inverse.
= 2 x /3		
= 2/3.		

In the conventional notation this is: 1/3 / 1/2 = 2 x 1/3 = 2/3, or

$$\frac{\frac{1}{3}}{\frac{1}{2}} = \frac{1}{3} \times \frac{2}{1} = \frac{2}{3}$$

F. The Next Step in Number Structure—the Decimal Point

Now that we know how to manipulate fractions, let's look at a number in expanded notation:

435 = 400 + 30 + 5
= 4 x 100 + 3 x 10 + 5 x 1.

If we multiply this number by ten, we get:

$$10 \times (4 \times 100 + 3 \times 10 + 5 \times 1) = 4 \times 100 \times 10 + 3 \times 10 \times 10 + 5 \times 10$$
$$= 4 \times 1000 + 3 \times 100 + 5 \times 10$$
$$= 4350.$$

Graphically, we just concatenate a zero to "everything." Each digit has been shifted over to the next place position. To divide 4350 by ten we would shift all the digits back to their original positions.

Let us look at fractions.

$$6/10 + 7/100 + 8/1000 = 6/10 \times 1 + 7/100 \times 1 + 8/1000 \times 1$$
$$= 6/10 \times 100/100 + 7/100 \times 10/10 + 8/1000 \times 1/1$$
$$= 600/1000 + 70/1000 + 8/1000$$
$$= (600 + 70 + 8) / 1000$$
$$= 678 / 1000$$

If we multiply this number by 10, we get $600/1000 \times 10 + 70/1000 \times 10 + 8/1000 \times 10$.

Since $/1000 = /100 \times /10$ and $10 \times /10 = 1$, we get

$$600/100 + 70/100 + 8/100 = 6 \times 100/100 + 70/100 + 8/100$$
$$= \mathbf{6 + 78/100}$$

Let us rename 6/10 as .6, 7/100 as .07, and 8/1000 as .008. If we lined these up and added them we would get:

$$\begin{array}{l} .6 \\ .07 \\ \underline{.008} \\ .678 \end{array}$$

Even though we will learn that this example is correct, currently we are just applying plausible patterns. Now consider:

$$435 + 678 / 1000$$

If we multiply this number by ten we get

$$10 \times (435 + 678 / 1000) = 4350 + \mathbf{6 + 78 / 100}$$
$$= 4356 + 78 / 100$$

Now write 435 + 678 / 1000 as 435.678. If we apply the pattern of shifting the digits one place to the left, we get 4356.78 Dividing this number by ten, we should shift the numbers to the right: 435.678.

It appears that the decimal point may have been discovered by simple pattern recognition. What happens when we multiply 52.91 x 16.8?.

Because 52.91 x 16.8 = 5291/100 x 168/10
= 5691 x 168/ (100 x10)
= 5691 x 168 /1000
= 888888/1000
= 888.888,

it appears that we add up the number of digits to the right of each decimal and that gives us the position in the answer.

G. Conclusions

Addition and multiplication has been taught by many as a rote memory process or what we call 'learning our number facts." Long multiplication is taught by teaching children the process without explaining the process. We have learned that understanding the Distributive Rule explains that process. Multiplication was learned by a pattern recognition process, but that process required no understanding. Without understanding it is difficult to make discovery.

We were then perplexed as to why fractions became so difficult for the general population to understand. There was no base upon which to build the knowledge. Comprehending the addition of fractions requires understanding the Distributive Rule. Multiplying and dividing fractions requires understanding the definition of the fraction. Because fractions and negative numbers follow similar patterns, a better understanding of negative numbers would simplify the understanding of fractions. Look at the comparisons:

	Negative Numbers	Fractions	
1.	2 + -2 = 0	2 x / 2 = 1	Definition
2.	2 = 0 – -2 = - -2	2 = 1 / / 2 = / /2	
3.	-2 = 0 – 2 = - 2	/2 = 1 / 2	
4.	5 – -2 = 5 + 2 + -2 - -2 = 7	/3 + /2 = 2 x /2 x /3 + 3 x /3 x /2 = 2 x /6 + 3 x /6 = 5/6	
5.	-2 = 2 x –1	14 / 7 = 2 x 7 / 7 = 2	
6.	Adding a negative number gives an answer less than the number to which we added it.	Multiplying by a fraction gives an answer less than the number by which we multiplied it.	